University of California, Berkeley
Physics 110A Spring 2003 Section 2 (*Strovink*)

Solution Set 9 (Compiled by Uday Varadarajan)

1. This problem is most easily solved by noting that the circuit described can be understood as a pair of voltage dividers connected in parallel to the EMF. A voltage divider is just a simple circuit in which a pair of circuit elements (i.e. either inductors, capacitors, or resistors) are connected in series from a power supply to ground, with a lead attached between them where a voltage is measured. If we complexify the voltages and currents, and take the impedences (these are complex generalizations of resistances defined below, just think of them as resistors for now) of the two elements to be Z_1 and Z_2, then the voltage measured by an ideal voltmeter which draws no current is just the real part of,
$$\mathcal{E}_{\text{out}} = \frac{Z_2}{Z_1 + Z_2}\mathcal{E}_{\text{in}}. \tag{1}$$
In our case, we just have a pair of voltage dividers where the role of Z_1 and Z_2 are interchanged between the two. Thus, the voltage difference between the two cases will just be the real part of,
$$\mathcal{E}_{\text{out}} = \frac{Z_2 - Z_1}{Z_1 + Z_2}\mathcal{E}_{\text{in}}. \tag{2}$$
Now all that remains is to explain what the impedences of a capacitor and an inductor are. This is easy to compute - the voltage drop across a resistor, inductor, and capacitor are given by IR, $L\frac{dI}{dt}$, and Q/C. If the current is sinusoidal, then the easiest way to analyze the system is to complexify everything resulting in complex currents and then take the real part at the end of the day. That is, take the current to be $I(t) = \Re I_0 e^{i\omega t}$. Then, one finds that the complexified voltage drops across all the above circuit elements simplify to expressions that are complex generalizations of resistances, $I(t)R$, $I(t)(iL\omega)$, and $I(t)\frac{1}{i\omega C}$. Thus, we see that the impedence of a capacitor and inductor are given by,
$$Z_C = \frac{1}{i\omega C} \qquad Z_L = i\omega L. \tag{3}$$
To see that this makes sense, note that if $\omega = 0$, we expect that no current should flow across a capacitor while current should flow without any resistance through an inductor, which is indeed the case. Plugging these into the above expression, we find,
$$\mathcal{E}_{\text{out}} = \Re\left(\frac{\frac{1}{i\omega C} - iL\omega}{\frac{1}{i\omega C} + iL\omega}\right) = \left(\frac{1+\omega^2 LC}{1-\omega^2 LC}\right)\mathcal{E}_{\text{in}}. \tag{4}$$

2. (a) To 0^{th} order, this varying current means that we have a varying charge on the plates of the capacitor given by $Q(t) = \frac{I_0}{\omega}\sin\omega t$. This leads to a uniform electric field in the gap given by Gauss's Law to be, $E_z(t) = \frac{Q(t)}{\epsilon_0 \pi b^2} = \frac{I_0}{\epsilon_0 \omega \pi b^2}\sin\omega t$.

   (b) The varying electric field induces a varying magnetic field,
$$\oint \vec{B}\cdot d\vec{l} = 2\pi s B_\phi(s,t) = \mu_0\epsilon_0 \frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{a} = \frac{\mu_0 I_0 s^2}{\omega b^2}\cos\omega t \Rightarrow B_\phi(s,t) = \frac{\mu_0 I_0 s}{2\pi\omega b^2}\cos\omega t. \tag{5}$$

   (c) To first appproximation, this varying magnetic field will induce an additional variation of the z component of the electric field in the cavity. To see this, consider a loop formed by a line segment between the two plates that then just proceeds radially outward along both plates and reconnects in a line segment outside the capacitor. As we have neglected fringing fields and as the electric field in the conducting plates vanishes, an integral over this loop just reduces to an integral over the line segment inside the capacitor. Further, the surface bounded by this loop proceeds radially outward from the line segment, that is, it is at constant angle ϕ and just spans a radial distance from s to b and height from 0 to d. Note that since Faraday's Law can only tell us about the curl of $\vec{E}$, it misses the 0^{th} order contribution to $\vec{E}$, so we find,

$$\oint \vec{E}^1 \cdot d\vec{l} = E^1_z(s,t)d = -d\frac{\partial}{\partial t}\int_s^b ds \frac{\mu_0 I_0 s}{2\pi\omega b^2}\cos\omega t \Rightarrow E^1_z(s,t) = \frac{\mu_0 I_0 (b^2 - s^2)}{4\pi b^2}\sin\omega t. \tag{6}$$

3. For simplicity, suppose the axis of rotation is the $\hat{\mathbf{z}}$ axis, and the electric field points in the $\hat{\mathbf{x}}$ direction. Then, due to the rotation, the normal vector to the disk which is bounded by a loop of radius b going around the length of the toroid is varying as,

$$\hat{\mathbf{n}} = (\cos\omega t, \sin\omega t, 0). \tag{7}$$

This results in a varying electric flux through the loop,

$$\Phi_E = \pi b^2 \hat{\mathbf{n}} \cdot \vec{E}_0 = \pi b^2 E_0 \hat{\mathbf{n}} \cdot \hat{\mathbf{x}} = \pi b^2 E_0 \cos\omega t \tag{8}$$

Neglecting the width of the toroid, this varying flux gives rise to a nearly uniform, time-varying magnetic field through the toroid,

$$\int \vec{B} \cdot d\vec{l} = 2\pi b B_\phi(t) = \mu_0\epsilon_0 \frac{\partial \Phi_E}{\partial t} = -\mu_0\epsilon_0\omega\pi b^2 E_0 \sin\omega t \Rightarrow B_\phi(t) = -\frac{1}{2}\mu_0\epsilon_0\omega b E_0 \sin\omega t. \tag{9}$$

The varying magnetic field results in a varying mangnetic flux through the turns of the toroid, giving rise to an EMF,

$$\mathcal{E} = -\frac{\partial \Phi_B}{\partial t} = \frac{1}{2}\mu_0\epsilon_0 n\omega^2\pi b a^2 E_0 \cos\omega t. \tag{10}$$

We can compute the current resulting from this emf using the inductance of the toroid. This can be easily approximated using the fact that since $b \gg a$, the inductance of the toroid is essentially the same as a solenoid of length $2\pi b$ with $n/(2\pi b)$ turns per unit length, for which,

$$\Phi_B = n(\mu_0(n/(2\pi b)I)\pi a^2 = (\frac{1}{2b}\mu_0 n^2 a^2)I = LI \Rightarrow L = \frac{1}{2b}\mu_0 n^2 a^2. \tag{11}$$

Thus, we can compute the induced current by,

$$\mathcal{E} = \frac{1}{2}\mu_0\epsilon_0 n\omega^2\pi b a^2 E_0 \cos\omega t = L\frac{dI}{dt} \Rightarrow |I_{max}| = \frac{1}{2\omega L}\mu_0 n\epsilon_0\omega^2\pi b a^2 E_0 = \frac{\pi b^2\epsilon_0\omega E_0}{n}. \tag{12}$$

4. When written in terms of potentials, Gauss's Law becomes, (Griffiths 10.4), in Coulomb Gauge,

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{1}{\epsilon_0}\rho \Rightarrow \nabla^2 V = -\frac{1}{\epsilon_0}\rho \tag{13}$$

precisely Poisson's equation. Thus, we see that the scalar potential of a point particle in this gauge is exactly the Coulomb potential in electrostatics, hence the name. The equation for the vector potential in Coulomb gauge is just Griffiths 10.5,

$$\left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) = -\left(\mu_0\epsilon_0\frac{\partial(-\nabla V)}{\partial t} + \mu_0\vec{J}\right). \tag{14}$$

Since $-\nabla V$ is the electric field if we were doing electrostatics, we can interpret this result as telling us that such a varying "electric field" can be associated with an effective current.

5. Suppose that after a gauge transformation,

$$\vec{A}' = \vec{A} + \nabla\lambda \tag{15}$$

$$V' = V - \frac{\partial\lambda}{\partial t}, \tag{16}$$

we have $V' = 0$, and in this gauge, we have, $\vec{E} = -\frac{\partial \vec{A}'}{\partial t}$. Thus, we see that $\vec{A}' = \int \vec{E} dt$, up to a vector field which is time independent. This ambiguity reflects the residual gauge freedom we have to make gauge transformations which do not change $V' = 0$, i.e., gauge transformations which involve time independent $\lambda(x)$, $\frac{\partial \lambda}{\partial t} = 0$. This precisely adds to $\vec{A}$ a time independent vector field $\nabla \lambda$.

6. (a) Since we are just putting in a constant current, this corresponds to a linear build up of charge in times, $Q(t) = It$, and as the charge is distributed uniformly, the resulting electric field is just,

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{z}} = \frac{It}{\pi a^2 \epsilon_0}\hat{\mathbf{z}}. \tag{17}$$

This varying electric field results in a varying magnetic field circulating in the gap via Ampere/Maxwell's law,

$$B_\phi(s,t) 2\pi s = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\frac{It}{\pi a^2 \epsilon_0}\pi s^2\right) \Rightarrow B_\phi(s,t) = \frac{\mu_0 I s}{2\pi a^2}. \tag{18}$$

(b) To compute the energy density, we just use,

$$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + (s/2)^2\right], \tag{19}$$

while computing the Poynting vector just requires us to remember that $\hat{\mathbf{z}} \times \hat{\boldsymbol{\phi}} = -\hat{\mathbf{s}}$, so we find that it points inwards,

$$\vec{P} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = -\frac{I^2 s t}{2\pi^2 \epsilon_0 a^4}\hat{\mathbf{s}}. \tag{20}$$

Now, we can check that this satisfies energy conservation, Griffiths 8.14,

$$\frac{\partial u_{em}}{\partial t} = \frac{I^2 t}{\pi^2 \epsilon_0 a^4} = -\nabla \cdot \vec{P} = \frac{I^2 t}{2\pi^2 \epsilon_0 a^4}\nabla \cdot (s\hat{\mathbf{s}}) = \frac{I^2 t}{\pi^2 \epsilon_0 a^4}. \tag{21}$$

(c) We can determine the total energy in the gap out to a radius b by just integrating,

$$U_{em} = \int u_{em} d\tau = \int u_{em} w 2\pi s ds = \frac{w \mu_0 I^2}{\pi a^4}\int_0^b \left[(ct)^2 + (s/2)^2\right] s ds = \frac{w \mu_0 I^2 b^2}{2\pi a^4}\left[(ct)^2 + \frac{b^2}{8}\right]. \tag{22}$$

To get the power flowing into the gap, we just integrate over a surface at radius b and then compare to the rate at which the energy is increasing to find,

$$P_{in} = -\int \vec{P} \cdot d\vec{a} = \frac{I^2 w t b^2}{\pi \epsilon_0 a^4} = \frac{dU_{em}}{dt} = \frac{w \mu_0 I^2 b^2}{2\pi a^4} 2c^2 t = \frac{I^2 w t b^2}{\pi \epsilon_0 a^4}. \tag{23}$$

7. (a) Consider a light beam composed of n photons/m^3. Since each photon has an energy E, the energy density is just $u = En$. The energy flux is the energy density multiplied by the velocity of each photon, and is therefore $uc = Enc$.

(b) Now, if each photon carries a momentum p, then the momentum density of the beam is just is the momentum of each photon multiplied by the number density of photons, $\mathcal{P} = np$. Since, for a photon, $m = 0$, we have $E = pc$, so we have, $\mathcal{P} = nE/c$. Now, as the magnitude of the Poynting vector is related to the energy flux, using the result from part (a), $\mathcal{P} = nE/c = (Enc)/c^2 = S/c^2$.

8. The idea here is that the due to the presence of an electric field associated with the charges as well as the magnetic field due to the current through the solenoid, we obtain a non-zero Poynting vector which circulates around the disk. Using the result of problem 7, this is to be interpreted as a non-zero momenum density that circulates around the disk, before the current is turned off. Thus, this leads to a non-vanishing angular momentum in the system, stored in the electromagnetic fields. It is this angular momentum that is transformed into mechanical angular momentum as the current decays.